Pure Core 4 Past Paper Questions

Taken from MAP2, MAP3

Pure 2 June 2001

6 (a) Given that $\tan x \neq 1$, show that

$$\frac{\cos 2x}{\cos x - \sin x} \equiv \cos x + \sin x.$$ *(3 marks)*

(b) By expressing $\cos x + \sin x$ in the form $R\sin(x + a)$, solve, for $0° \leqslant x \leqslant 360°$,

$$\frac{\cos 2x}{\cos x - \sin x} = \frac{1}{2}.$$ *(5 marks)*

Pure 2 January 2002

4 (a) Prove that

$$\frac{2\tan x}{1 + \tan^2 x} \equiv \sin 2x.$$ *(4 marks)*

(b) Hence or otherwise find the exact value of $\tan 15°$ in the form $a + b\sqrt{3}$, where a and b are integers. *(5 marks)*

5 (a) Given that $\sin\alpha = \frac{12}{13}$, where α is an obtuse angle, find the exact value of $\cos\alpha$. *(2 marks)*

(b) Given also that $\cos\beta = \frac{4}{5}$, where β is an acute angle, find the exact value of $\sin(\alpha + \beta)$.

(3 marks)

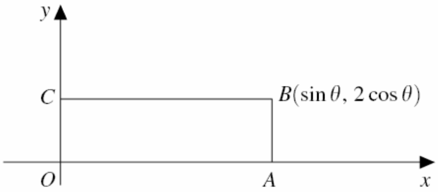
Pure 2 June 2002

8 (a) Show that $\dfrac{\cot^2\theta}{1 + \cot^2\theta} \equiv \cos^2\theta.$ *(3 marks)*

(b) Hence solve $\dfrac{\cot^2\theta}{1 + \cot^2\theta} = 2\sin 2\theta$ for $0° \leqslant \theta \leqslant 360°$. *(6 marks)*

Pure 2 January 2004

4



The diagram shows a rectangle $OABC$ in which B has coordinates $(\sin\theta, 2\cos\theta)$, where $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

The perimeter of the rectangle is of length L.

(a) (i) Write down the length L in terms of θ. *(1 mark)*

(ii) Hence obtain an expression for L in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0 \leqslant \alpha \leqslant \frac{\pi}{2}$.
Give your answer for α to three decimal places. *(4 marks)*

(b) Given that θ varies between 0 and $\frac{\pi}{2}$:

(i) write down the maximum value of L; *(1 mark)*

(ii) find the value of θ, to two decimal places, for which L is maximum. *(2 marks)*

Pure 2 June 2003

3 (a) Show that $\tan(45° + \theta) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$. *(2 marks)*

(b) Hence obtain the exact value of tan 105° in the form $a + b\sqrt{3}$, where a and b are integers to be found. *(4 marks)*

Pure 2 January 2004

3 (a) Find the value of $\tan^{-1} 2.4$, giving your answer in radians to three decimal places. *(1 mark)*

(b) Express $10\sin\theta + 24\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. *(3 marks)*

(c) Hence:

(i) write down the maximum value of $10\sin\theta + 24\cos\theta$; *(1 mark)*

(ii) find a value of θ at which this maximum value occurs. *(2 marks)*

Pure 2 June 2004

2 (a) Show that $\sin(\alpha + \beta) + \sin(\alpha - \beta) \equiv 2\sin\alpha\cos\beta$. *(2 marks)*

(b) (i) Express $2\sin 8x\cos 2x$ in the form $\sin A + \sin B$. *(2 marks)*

(ii) Hence find $\int 6\sin 8x\cos 2x\,dx$. *(3 marks)*

Pure 3 June 2001

1 (a) Sketch the curve given by the parametric equations

$$x = 4t^2, \; y = 8t \qquad \text{for } t \geqslant 0 .$$ *(1 mark)*

(b) Find $\dfrac{dy}{dx}$ in terms of t. *(2 marks)*

(c) Hence find the equation of the tangent at the point where $t = 0.5$. *(3 marks)*

3 The rate of increase in population of bacteria is proportional to the size of the population that exists at any particular time.

(a) Explain briefly why this situation can be modelled by a differential equation of the form

$$\frac{dP}{dt} = kP ,$$

where P is the size of the population, k is a constant and t is the time in minutes measured from a given starting time. *(2 marks)*

(b) (i) At time $t = 0$, the population of bacteria of type A is 1000. After 30 minutes, this population is 2000.
Solve the differential equation, stating the value of k in the form $q \ln 2$, where q is a rational number to be found. *(4 marks)*

(ii) As the population of type A increases, the population of another type of bacteria, B, decreases. The population, Q, of the type B bacteria at time t minutes is modelled by

$$Q = 5000\, e^{-0.05t} .$$

Find, to the nearest minute, the value of t when the populations of bacteria of types A and B are the same. *(4 marks)*

6 The line l_1 has equation

$$\mathbf{r} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + s \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}.$$

This line intersects the plane $3x - y - 2z = 9$.

(a) Find the acute angle between the line l_1 and this plane. *(5 marks)*

(b) The line l_2 passes through the points $A(3, -2, 4)$ and $B(5, 1, 7)$.
Show that the lines l_1 and l_2 intersect. *(5 marks)*

7 The function f is given by

$$f(x) = \frac{9}{(1+2x)(4-x)}\ .$$

(a) Express $f(x)$ in partial fractions. *(3 marks)*

(b) (i) Show that the first three terms in the expansion of

$$\frac{1}{4-x}$$

in ascending powers of x are

$$\frac{1}{4} + \frac{x}{16} + \frac{x^2}{64}\,.$$ *(3 marks)*

(ii) Obtain a similar expansion for

$$\frac{1}{1+2x}\ .$$ *(2 marks)*

(iii) Hence, or otherwise, obtain the first three terms in the expansion of $f(x)$ in ascending powers of x. *(2 marks)*

(iv) Find the range of values of x for which the expansion of $f(x)$ in ascending powers of x is valid. *(2 marks)*

(c) (i) Find $\int f(x)\,dx$. *(2 marks)*

(ii) Hence find, to two significant figures, the error in using the expansion of $f(x)$ up to the term in x^2 to evaluate

$$\int_0^{0.25} f(x)\,dx.$$ *(4 marks)*

Pure 3 January 2002

2 A curve is given by the parametric equations

$$x = 2t + 3\,, \qquad y = \frac{2}{t}\,.$$

Find the equation of the normal at the point on the curve where $t = 2$. *(6 marks)*

4 (a) A car has a value of £15 000 when new and a value of £11 000 exactly 2 years later. The value of the car as it depreciates can be modelled by

$$V = Pe^{-kt}\,,$$

where £V is the value of the car t years after it is sold as new, and P and k are constants.

(i) State the value of P. *(1 mark)*

(ii) Find the value of k, giving your answer to three decimal places. *(3 marks)*

(b) Another car depreciates in value according to the model

$$W = 18000\,e^{-0.175t}\,,$$

where £W is its value t years after it is sold as new.

Assuming that both cars were sold as new on 1st January 2000, calculate the year during which they will have depreciated to the same value. *(4 marks)*

7 A rectangular water tank rests on a horizontal surface. The tank is emptied in such a way that the depth of water decreases at a rate which is proportional to the square root of the depth of the water.

The depth of the water is h metres at time t hours. Initially there is a depth of 1 metre of water in the tank. The depth of water is 0.5 metres after 2 hours.

(a) (i) Write down a differential equation for h. *(2 marks)*

(ii) Hence show that $2\sqrt{h} = 2 - kt$, where k is a constant. *(3 marks)*

(iii) Find the value of k giving your answer to three decimal places. *(2 marks)*

(b) Find how long it will take to empty the tank completely, giving your answer to the nearest minute. *(2 marks)*

1 A curve is given by the parametric equations

$$x = 1 - t^2 \ , \quad y = 2t \ .$$

(a) Find $\frac{dy}{dx}$ in terms of t. *(2 marks)*

(b) Hence find the equation of the normal to the curve at the point where $t = 3$. *(4 marks)*

2 (a) Express $\frac{4 - x}{(1 - x)(2 + x)}$ in the form $\frac{A}{1 - x} + \frac{B}{2 + x}$. *(3 marks)*

(b) (i) Show that the first **three** terms in the expansion of

$$\frac{1}{2 + x}$$

in ascending powers of x are $\frac{1}{2} - \frac{x}{4} + \frac{x^2}{8}$. *(3 marks)*

(ii) Obtain also the first **three** terms in the expansion of

$$\frac{1}{1 - x}$$

in ascending powers of x. *(2 marks)*

(c) Hence, or otherwise, obtain the first **three** terms in the expansion of

$$\frac{4 - x}{(1 - x)(2 + x)}$$

in ascending powers of x. *(2 marks)*

3 A curve has equation $\frac{x^2}{9} + \frac{y^2}{25} = 1$.

(a) Find the y–coordinates of the two points on the curve at which the x–coordinate is 2. *(2 marks)*

(b) Find the values of the gradient of the curve at these two points, giving your answers to two significant figures. *(4 marks)*

4 The decay of a radioactive substance can be modelled by the equation

$$m = m_0 \, e^{-kt},$$

where m grams is the mass at time t years, m_0 grams is the initial mass, and k is a constant.

(a) The time taken for a sample of the radioactive substance strontium 90 to decay to half of its initial mass is 28 years. Show that the value of k is approximately 0.024755. *(4 marks)*

(b) A sample of strontium 90 has a mass of 1 gram. Assuming this mass has resulted from radioactive decay, use the model to find the mass this sample would have had 100 years ago. Give your answer to three significant figures. *(2 marks)*

5 (a) Solve the differential equation $\frac{dy}{dx} = \frac{1}{y^2}$

giving the general solution for y in terms of x. *(4 marks)*

(b) Find the particular solution of this differential equation for which $y = -1$ when $x = 1$. *(2 marks)*

8 The line l_1 has equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$.

The line l_2 has equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 2 \end{bmatrix} + s \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$.

(a) Show that the lines l_1 and l_2 intersect and find the coordinates of their point of intersection. *(5 marks)*

(b) (i) Show that the vector $\begin{bmatrix} 1 \\ 11 \\ -16 \end{bmatrix}$ is perpendicular to both l_1 and l_2. *(2 marks)*

Pure 3 January 2003

1 (a) Obtain the binomial expansion of $(1+x)^{\frac{1}{2}}$ as far as the term in x^2. *(2 marks)*

(b) (i) Hence, or otherwise, find the series expansion of $(4+2x)^{\frac{1}{2}}$ as far as the term in x^2. *(3 marks)*

(ii) Find the range of values of x for which this expansion is valid. *(1 mark)*

2 A curve is defined by the parametric equations

$$x = 3\sin t \qquad \text{and} \qquad y = \cos t.$$

(a) Show that, at the point P where $t = \frac{\pi}{4}$, the gradient of the curve is $-\frac{1}{3}$. *(3 marks)*

(b) Find the equation of the tangent to the curve at the point P, giving your answer in the form $y = mx + c$. *(4 marks)*

3 (a) (i) Show that

$$\frac{x^2}{x^2-16} = 1 + \frac{16}{x^2-16}.$$ *(1 mark)*

(ii) Express

$$\frac{16}{x^2-16} \quad \text{in the form} \quad \frac{A}{x-4} + \frac{B}{x+4}.$$ *(2 marks)*

(b) Hence find

$$\int_5^8 \frac{x^2}{x^2-16}\,dx,$$

giving your answer in the form $p + q \ln r$. *(4 marks)*

7 A group of students are researching the rate at which ice thickens on a frozen pond. They have experimental evidence that when the air temperature is $-T\,°C$, the ice thickens at a rate

$$\frac{T}{14\,000x} \text{ cm s}^{-1},$$

where x cm is the thickness of the ice that has already formed.

On a particular winter day the air temperature is constant at $-7°C$. At 12.00 noon the students note that the ice is 2 cm thick. Time t seconds later the thickness of the ice is x cm.

(a) Show that

$$\frac{dx}{dt} = \frac{1}{2000x}.$$ *(1 mark)*

(b) Solve the differential equation and hence find the time when the students predict the ice will be 3 cm thick. *(5 marks)*

Pure 3 June 2003

2 A curve is given by the parametric equations

$$x = 3t - 1, \qquad y = \frac{1}{t}.$$

(a) Find $\frac{dy}{dx}$ in terms of t. *(2 marks)*

(b) Hence find the equation of the normal to the curve at the point where $t = 1$. *(4 marks)*

4 Observations were made of the number of bacteria in a certain specimen. The number N present after t minutes is modelled by the formula

$$N = Ac^t,$$

where A and c are constants.

Initially there are 1000 bacteria in the specimen.

(a) Write down the value of A. *(1 mark)*

(b) Given that there are 12 000 bacteria after 60 minutes, show that the value of c is 1.0423 to four decimal places. *(3 marks)*

(c) (i) Express t in terms of N. *(3 marks)*

(ii) Calculate, to the nearest minute, the time taken for the number of bacteria to increase from one thousand to one million. *(1 mark)*

Pure 3 January 2004

1 A curve is given by the parametric equations

$$x = 3t^2, \qquad y = 6t.$$

(a) (i) Find $\dfrac{dy}{dx}$ in terms of t. *(2 marks)*

(ii) Find the gradient of the curve at the point where $t = \frac{1}{2}$. *(1 mark)*

(b) (i) Find the equation of the curve in the form $x = f(y)$. *(2 marks)*

(ii) Find $\dfrac{dx}{dy}$ in terms of y and hence verify your answer to part (a)(ii). *(4 marks)*

3 A microbiologist is studying the growth of populations of simple organisms.

For one such organism, the model proposed is

$$P = 100 - 50e^{-\frac{1}{4}t},$$

where P is the population after t minutes.

(a) Write down:

(i) the initial value of the population; *(1 mark)*

(ii) the value which the population approaches as t becomes large. *(1 mark)*

(b) Find the time at which the population will have a value of 75, giving your answer to two significant figures. *(4 marks)*

4 (a) Express $\dfrac{8+3x}{(1+3x)(2-x)}$ in the form $\dfrac{A}{1+3x}+\dfrac{B}{2-x}$. *(3 marks)*

(b) Obtain the first three terms in the expansion of $\dfrac{1}{1+3x}$ in ascending powers of x. *(2 marks)*

(c) Show that the first three terms in the expansion of $\dfrac{1}{2-x}$ in ascending powers of x are $\dfrac{1}{2}+\dfrac{x}{4}+\dfrac{x^2}{8}$. *(3 marks)*

(d) Hence, or otherwise, obtain the first three terms in the expansion of $\dfrac{8+3x}{(1+3x)(2-x)}$ in ascending powers of x. *(3 marks)*

(e) State the range of values of x for which the expansion in part (d) is valid. *(2 marks)*

6 The speed v m s^{-1} of a pebble falling through still water after t seconds can be modelled by the differential equation

$$\frac{dv}{dt} = 10 - 5v.$$

A pebble is placed carefully on the surface of the water at time $t = 0$ and begins to sink.

(a) Show that $t = \dfrac{1}{5}\ln\left(\dfrac{2}{2-v}\right)$. *(6 marks)*

(b) Use the model to find the speed of the pebble after 0.5 seconds, giving your answer to two significant figures. *(3 marks)*

7 The points A and B have coordinates $(3, -1, 2)$ and $(5, 3, -2)$ respectively.

(a) (i) Find the distance between A and B. *(2 marks)*

(ii) Find the coordinates of M, where M is the mid-point of AB. *(1 mark)*

(b) The point C has coordinates $(8, -2, -1)$.

Show that $\overrightarrow{CM}$ is perpendicular to $\overrightarrow{AB}$. *(2 marks)*

Pure 3 June 2004

1 A curve is given by the parametric equations

$$x = 2t - 1, \quad y = \frac{1}{2t}.$$

(a) Find $\dfrac{dy}{dx}$ in terms of t. *(2 marks)*

(b) Find the equation of the normal to the curve at the point where $t = 1$. *(4 marks)*

2 (a) Obtain the binomial expansion of $(1+x)^{\frac{1}{3}}$ as far as the term in x^2. *(2 marks)*

(b) Hence, or otherwise, find the series expansion of $(8+4x)^{\frac{1}{3}}$ as far as the term in x^2. *(3 marks)*

3 (a) Express $\dfrac{30}{(x+4)(7-2x)}$ in the form $\dfrac{A}{x+4}+\dfrac{B}{7-2x}$. *(3 marks)*

(b) Hence find

$$\int_0^3 \frac{30}{(x+4)(7-2x)}\,dx,$$

giving your answer in the form $p \ln q$, where p and q are rational numbers. *(5 marks)*

4 A curve is given by the equation $9(y+2)^2 = 5 + 4(x-1)^2$.

(a) Find the coordinates of the two points on the curve where $x = 2$. *(3 marks)*

(b) Find the gradient of the curve at each of these points. *(5 marks)*

7 Initially there are 2000 fish in a lake. The number of fish, x, at time t months later is modelled by the differential equation

$$\frac{dx}{dt} = x(1-kt),$$

where k is a constant.

(a) Solve this differential equation to show that

$$x = 2000e^{t-\frac{1}{2}kt^2}.$$ *(6 marks)*

(b) After 12 months the number of fish is again 2000. Find the value of k. *(3 marks)*

8 (a) Find the vector equation of the line l_1, which passes through the points $A(3, -1, 2)$ and $B(2, 0, 2)$. *(2 marks)*

(b) The line l_2 has vector equation $\mathbf{r} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

Show that the lines l_1 and l_2 intersect and find the coordinates of their point of intersection. *(4 marks)*

(c) Show that the point $C(9, 1, -6)$ lies on the line l_2. *(2 marks)*

(d) Find the coordinates of the point D on l_1 such that CD is perpendicular to l_1. *(4 marks)*